The Logical Relation of Consequence

Basil Evangelidis
Department of History and Philosophy of Science,
University of Athens
University Campus, Ano Ilissia, 15771, Athens, Greece
E-mail: vasevang@ieee.org

Abstract:

The present endeavour aims at the clarification of the concept of the logical consequence. Initially we investigate the question: How was the concept of logical consequence discovered by the medieval philosophers? Which ancient philosophical foundations were necessary for the discovery of the logical relation of consequence and which explicit medieval contributions, such as the notion of the formality (formal validity), led to its discovery. Secondly we discuss which developments of the modern philosophy effected the turn from the medieval concept of logical consequence to its most recent conceptions, such as the semantic, syntactic, axiomatic and natural deductive approaches? Thirdly we examine which are the similarities and the differences between the logical concepts of consequence, inference, implication and entailment? Furthermore, we ask what kind of relation signifies the concept of the logical consequence? That is to say, which is the analytic definition of the consequence relation R between the premises $p_1, p_2..p_n$ and the conclusion c of a formally valid argument? Finally, we focus on the respective answers given through the developments in proof theory by David Hilbert and Gerhard Gentzen.

Keywords: medieval philosophy, consequence, validity, formality, inference, implication, proof theory, semantic, syntactic, natural deduction

1. How was the concept of logical consequence discovered by the medieval philosophers?

The Latin term “consequentia” was introduced by Boethius, in his *Commentary on ‘On Interpretation’* (2, 109-10). At a first stage, Aristotle in the *De Interpretatione* (5, 17a9-10 and 20-22), had used the concepts “(non-)connected,” “conjunction” or “connective” for the expression of the relation between parts of a composite statement. Furthermore, the theory of enthymemata of the *Topics* and the theory of inference of the *Prior Analytics* of Aristotle, the five indemonstrable inference schemes of Chrysippus, and the books *De Differentiis Topicis* and especially *De syllogismis hypotheticis* of Boethius presented the first elements of a theory of consequence. The revival of the Aristotelian logic in the twelfth century motivated medieval logicians to develop their own original logic, the so-called *logica modernorum,* which included the theory of properties of terms, the theory of consequences, the theory of insolubles and the theory of obligations. The beginnings of a consequence theory are found in medieval commentaries on Aristotle and Boethius, in discussions of fallacies, and in treatises on syncategoremata. However, the medieval theory of consequences reached a complete form in the fourteenth century, especially in the years 1300-1340, with the works of Walter Burley (or Burleigh), William of Ockham and John Buridan.

A general theory of consequence, in the bases of the ancient assertoric and alethic modal syllogistic, was set up by Garland the Computist, Peter Abelard, the anonymous authors of *Logica Modernorum*, William of Sherwood, Lambert of Auxerre, Peter of Spain, and Burley's treatises on consequences and logical obligations. Apart from these, there were the treatises on syncategoremata of William of Sherwood and Peter of Spain, along with Robert Grosseteste's commentary on *Posterior Analytics* and the logical works of Roger Bacon, Robert Kilwardby, Robert Holkot, Albert the Great and Thomas Aquinas (Boh 1993).

Some arguments are perfect in the sense that their premises suffice by themselves for the establishment of their conclusions, they formulate a consequentia which is true *secundum complexionem*, that is to say, true in virtue of its formal structure, for instance: "Si omnis homo est animal et omne animal est animatum, omnis homo est animatus" ("If every man is an animal and every animal is animate, every man is animate"). No matter what replacements of the terms we may try, we get always a true consequentia, if the logical form is preserved.

Abelard's doctrine of consequence was based on the *De Differentiis Topicis* of Boethius, as Spade (1994) imparts. Abelard's rules of the relation of consequence are the following: "1. on the antecedent being posited, the consequent is posited; 2. on the consequent being destroyed, the antecedent is destroyed, thus: 'if there is man there is animal', 'if there is not animal there is not man'; 3. neither if the antecedent is posited, is the consequent destroyed, 4. nor if the antecedent is destroyed need the consequent be destroyed 5. or posited, just as, 6. neither if the consequent is destroyed is the antecedent posited, 7. nor if the same (the consequent) is posited is it (the antecedent) either posited 8. or removed" (Thomas 1961, x).

The formalisation of reasoning was placed as a first priority by the theory of consequence. The study of dependence relations between propositions should start with the general rule: from something true, something false never follows. With substitution-testing we are able to find out if an inference is valid and the consequence relation is satisfied. Only one fallacious example is sufficient for the invalidation of an inference. Formally, therefore, every logically valid inference, if it satisfies the consequence relation, can never be falsified by any particular application.

An inference is valid, namely, a consequence holds, when the consequent follows from the antecedent and the antecedent is incompatible with the opposite of the consequent. Therefore, if the antecedent is true and the consequent is false, the consequence is not valid. However, the relation of antecedent to consequent in a conditional proposition is different from the relation of premisses to conclusion in an inference or argument. As King (2001) points out, conditionals make statements while inferences do things with statements. The conditionals indicate truth and belong to the object-language, while the inferences indicate validity and belong to the meta-language.

Furthermore, the conditionals in medieval logic, together with conjunctive and disjunctive expressions, belong to the general rank of "compound sentence" (*propositio hypothetica*).

According to earlier writers (Burley, Ockham) formal consequences are valid because of the meanings of terms. However, Buridan defined formal validity in the manner of modern logic, as the validity of all instances of the same logical form. He defined formal validity exclusively in terms of uniform substitution. Buridan in his *Treatise* (I, 2) introduced the notion of the "cause of truth" of a proposition and insisted on the role of the distribution of a term of a sentence, which is the starting point of the substitution process. A substitutional account was also proposed, in the nineteenth century, by Bernard Bolzano. The truth of a sentence, according to Bolzano, depends on its relation to the set of the sentences that occur from the substitution of one of its terms: if every grammatically appropriate substitution of this certain term renders this sentence true, then the sentence is logically true. The terms that we allow to vary are called by Bolzano *variables* and the terms that we keep fixed are called *fixed* terms. The substitutions of terms that we permit are critical for the truth of a sentence (Etchemendy 1999). The substitutional initiative was criticised, but finally many logicians utilised both definitions, the one based on meaning and amounting to analyticity, the other substitutional and pertaining to logical validity. The Parisian tradition connected the notion of formal validity with truth-preservation under all substitutions of non-logical terms. The English tradition proposed a containment-principle, which implied that the understanding of the antecedent should contain the understanding of the consequent, as Klima (2016) explains.

Our research will consider as seminal two medieval works on consequence:

▪ Walter Burley's *De consequentiis*: Whereas Ockham's Logic is arranged traditionally around terms, propositions, and arguments, Burley's is organised around the general rules of consequences, by giving priority to propositional logic. Burley's argumentation on the compatibility and incompatibility rules of the antecedent and the consequent is indicative: whatever contradicts with the consequent contradicts with the antecedent, and whatever stands with the antecedent stands with the consequent.

▪ Jean Buridan's *Tractatus de consequentiis*: Buridan's *Treatise* covers both theory of conditionals and rules of inference. Buridan's semantics is token rather than type-based and it doesn't assume that a material consequence is not a logical one, as Archambault (2017) explains. The definition of logical consequence provided by Buridan (2015, 66) is:

> Now a consequence is a compound proposition (*propositio hypothetica*); for it is constituted from several propositions conjoined by the expression "if" or the expression "therefore" or something equivalent. For these expressions mean that of propositions conjoined by them one follows from the other; and they differ in that the expression "if" means that the proposition immediately following it is the antecedent and the other the consequent, but the expression "therefore" means the converse.

Buridan also distinguished formal and material consequence on the basis of their consisting terms. Material consequence consists of categorematic terms (subject and predicate), whereas formal consequence consists of syncategorematic terms (logical constants). Material consequences include enthymemes, examples and inductions.

> I say that when we speak of matter and form, by the matter of a proposition or consequence we mean the purely categorematic terms, namely, the subject and predicate, setting aside the syncategoremes attached to them by which they are conjoined or denied or distributed or given a certain kind of supposition; we say all the rest pertains to the form. So we say that the copulas of both simple subject-predicate and compound propositions pertain to the form, as do negations, [other] signs, the number of propositions and terms and the mutual relation of all these, and relations of anaphoric terms and modes of signifying pertaining to the quantity of the proposition, for example, whether discrete or general, and many other things that the attentive reader can recognise if they occur (Buridan 2015, 74).[1]

Ockham's *Summa Logicae,* at the beginning of his tract on topical rules, in the third main division of the third part of the work, offers a division of consequences: a) Consequences may be either factual or absolute. A factual consequence (*consequentia ut nunc*) is valid at one time and may be invalid at another. On the contrary, an absolute consequence (*consequentia simplex*), is always valid, regardless of the time element. This distinction is also made by Buridan and Burley. b) Furthermore, a consequence may be valid in virtue of an intrinsic means (*consequentia tenens per medium intrinsecum*) or in virtue of an extrinsic means. A consequence which is valid in virtue of an intrinsic means is in reality an enthymeme. Burley suggested also that simple consequence is divided in natural and accidental (Green-Pedersen 1980, 128, 69-70). The natural consequence is intrinsically understood, while the accidental extrinsically,[2] as Burley explained (1955, 61.6-10). c) A consequence may be formal or material. Formal consequence is valid because of a general rule, while material consequence is valid because of the truth or falsity of the propositions that enter the consequence: "A material consequence exists when it holds precisely because of the terms, and not because of some extrinsic means that precisely regards the general conditions of propositions. Such are the following: If a man is running, then God exists; Man is a donkey, therefore God does not exist" (*Summa Logicae*, partis 3, pars 3, c. I.). Moreover, Ockham states that whenever there is a case in which the consequent does not follow from the antecedent, the antecedent will follow from the consequent. However, as Lewis-Langford remark, this theorem holds only in material consequence or implication.

The rules of the theory of consequence, according to Ockham, are the following: 1) From something true, something false never follows (*Ex vero numquam sequitur falsum*). 2) From false propositions a true proposition may follow (*Ex falsis potest sequi verum*). 3) If a consequence is valid, the negative of its antecedent follows from the negative of its consequent. 4) Whatever follows from the consequent follows from the antecedent.[3] 5) If the antecedent follows from any proposition, the consequent follows from the same. 6) Whatever is consistent with the antecedent is consistent with the consequent. 7) Whatever is inconsistent with the consequent is inconsistent with the antecedent. 8) The contingent does not follow from the necessary. 9) The impossible does not follow from the possible. 10) Anything whatsoever follows from the impossible. 11) The necessary follows from anything whatsoever (*Summa Totius Logicae*, partis 3, pars 3, 37).

In a way similar with Ockham, Ralph Strode used the distinction between *illatio* and *intellectio* for the clarification of the difference between material and formal consequence, on the basis of the additional meaning or understanding (*intellectio*) that formal consequence involves. Around 1300, therefore, the theory of consequence had been developed under the recognition that modal notions, modal consequences and special modes have an analogous legitimate role, next to the alethic one (Boh 1993).

2. Which developments of the modern philosophy effected the turn from the medieval concept of logical consequence to its most recent conceptions?

The substitutional approach of Bolzano is only a necessary but not a sufficient requirement of logical truth in Tarski's philosophical logic. In fact, the substitutional method is liable to accidental results, because its notion of logical truth fails to persist through simple expansions of the language. For this reason, Tarski proposed the alternative criterion of the *satisfaction* of a sentential function. Tarski insisted on the untenability of new purely structural rules for the justification of logical consequence, such as the infinite induction between sentences conformable to a one-one correspondence with the natural numbers.

Tarski (1956, 414) summarised Carnap's definition of logical consequence with these words: "The sentence X follows logically from the sentences of the class K if and only if the class consisting of all the sentences of K and of the negation of X is contradictory." By modifying Carnap's definition of logical consequence, Tarski (1956, 417) defined logical consequence with the following words: "The sentence X follows logically from the sentences of the class K if and only if every model of the class K is also a model of the sentence X." To reconcile his definition with Carnap's, Tarski called a sentence or a class of sentences *contradictory* if it possesses no model, while he called a sentence or a class of sentences *analytical* if every sequence of objects is a model of it. Furthermore, a negation Y of a sentence X has as models those and only those sequences of objects which are not models of the sentence X.

Nevertheless, the application of the model-theoretic analysis of consequence by Tarski has been criticised by Etchemendy (1988; 1999), with the argument that it does not capture the intuitive, pre-theoretic and genuine notion of consequence. According to this critique, the syntactic proof-theoretic approach should not be considered as inferior to the model-theoretic. The modern successors of the theory of consequence can thus be found in the natural deduction systems of Jaśkowski and Gentzen (Hazen and Pelletier 2014). The correct use, the rule following and the inferential role are the natural grounds of meaning according to the semantic proof-theoretic approach proposed by Gentzen (1934/35; 1955). Natural deduction seeks normalisation or harmony between introduction and elimination, while being opposed to the axiomatisation systems, such as the ones of David Hilbert. The abandonment or, on the contrary, the maintenance of the meanings of the manipulated symbols stand as the hallmarks of the syntactic and the semantic approaches respectively.

3. Similarities and differences between the logical concepts of consequence, inference, implication and entailment

The two basic consisting parts of the medieval logic were the theory of the consequences and the theory of the properties of terms. The medieval philosophers would use the terms ‘inferentia’, ‘consecutio’ or ‘illatio’ equivalently, as a synonym of the term ‘consequentia’. On the other hand, William of Sherwood (1995, 74, 11-75, 4.) had distinguished four properties of terms in the following way:

> These properties are signification, supposition, copulation and appellation. Signification is the presentation of a form to the reason. Supposition is the ordering of one concept (intellectus) under another. Copulation is the ordering of one concept over another... But appellation is the present attribution of a term, i.e. the property by which what a term signifies can be predicated of something by means of the verb 'is'.

Moreover, by a careful comparison between the terms “consequence” and “inference”, it can be clearly shown that the logical consequence is based on an objective grounding relation, which equals to the justification of the validity of an inference. For instance, when John F. Kennedy wrote that television’s “revolutionary impact” would have far-reaching and lasting consequences for politics, he was predicting the future of the television. This example makes clear how effectively logical and causal consequence may relate to each other.

Relations may be classified according to asymmetry, transitiveness, and connexity, as Russell (1924) wrote. Causal relations are regarded as asymmetrical, non-reflexive, and transitive (Heil 2016); grounding is also an asymmetric, irreflexive and transitive relation, as Archambault (2019) holds. However, Aquinas insisted that a logical relation is a second intention, a thing that belongs merely to the intellect, such as the mental terms 'genus', 'species', 'noun', 'verb', 'case of a noun' etc.:

> Because relation has the weakest being of all the categories, some have thought that it belongs to second intentions (*intellectibus*). For the first things understood are the things outside the soul, to which the intellect is primarily directed, to understand them. But those intentions (*intentiones*) which are consequent on the manner of understanding are said to be secondarily understood. . . . So according to this thesis (*positio*) it would follow that relation is not among the things outside the soul but merely in the intellect, like the intention of genus and species and second (i.e. universal) substances (Thomas Aquinas, *De potentia* 7, 9, c).

Under this analysis, the relation of the consequence should be an objective but intellectual entity, a concept, a universal. Inference, implication and entailment should also belong to the general category of logical relations between propositions, where the consequence belongs.

According to Aristotle, deduction is a *necessary* inference from the premises to the conclusion. Boethius regarded as inference schemes those and only those logical formulae that begin with the functor "igitur" (therefore). On the other hand, the functors "si" (if) and "cum" (when) signified implication. Leibniz regarded as necessary the propositions which are true in all possible worlds.

Clearly, the concepts of the consequence, inference and entailment have strong liaisons with modal logic, while the notion of implication would seem more appropriate for semantic and linguistic quests. However, Hilbert and Ackermann (1955, 4) defined the logical constant of implication in a solely formalistic way:

> With "φ → ψ" (read "when φ, so ψ" or "from φ follows ψ"), we call a statement that indicates the implication formed from "φ" and "ψ" (in this order). "φ → ψ" is defined as follows: it is correct if "φ" is wrong, and also if "ψ" is correct. It is only wrong if "φ" is correct and "ψ" wrong. This makes the sense of " → " clearly determined.

From the formalistic definition of the basic connectives (logical constants), the correctness or falsehood of a linked statement is only dependent on the correctness or falsehood of the basic statements, but not at all from its content. We can therefore understand the linkages of statements as functions that assign one of the values "true" or "false" to the values "correct" or "false" of the linked statements. We therefore also call them *truth functions.*

4. What kind of relation signifies the concept of the logical consequence?

Truth preservation, proof-theory and quantification are three significant aspects of the philosophical problem of logical consequence:

The validity of an argument and the criterion of a correct analysis of logical consequence is the truth preservation from the premises to the conclusion. A valid argument cannot consist of true premises and false conclusion. This requirement is sufficient in the classical account and may be also commonly acceptable in a purely substitutional account, only with flexible adjustments of domain, interpretation and naming of elements, as Read (1995) insisted.

Truth preservation is the semantic aspect of Logic, where the correctness refers to the meaning of the symbols involved. On the other hand, proof-theory is the syntactic aspect of Logic, as the correctness of an application of a rule of inference through a series of steps, which depends entirely on logical form rather than meaning.

The great significance of the theory of consequence lies on the conception that the truth of a consequence depends on its formal, analytical, substitutional correctness. This formal correctness allows for quantification. Second order logic, higher order logic, set theory, the problem of the universals and the logic of relations are critical for a thorough understanding of the logic of consequence, since the relation of consequence pertains to a relation between propositions; more formally, between propositional functions. After the preliminary contribution of "the logic of relatives" by Peirce (1883, 1892) and the seminal analysis of the term "concept" as a function by Frege (1892), the term 'propositional function' occurs for the first time in Bertrand Russell's *Principles of Mathematics* (1903). By Tarski the respective term is called "sentential function."

Finally, the relation of the logical consequence between propositions ends up to the question of the grounding of a consequence, as Schnieder (2018) and Archambault (2019) propose. The grounding of a consequence relates to what Cicero and Boethius called *sedes argumenti,* the basis, seat, ground or region of an argument. The combination of a maximal proposition and a topical difference are the constituent parts of a locus, according to Boethius. Walter Burleigh (1955) also proposed that logical or dialectical topics can be either a maximal proposition or a difference of a maxim. Every good inference holds through a maximal proposition, i.e. a rule through which an inference holds. In modern Logic, Gentzen's (1934/5) rules of structural reasoning are considered as a successful grounding of logical consequence.

5. Proof theory and logic of relations

In the introduction of their joint work *Grundzüge der theoretischen Logik* (1928), Hilbert and Ackermann wrote that the idea of a mathematical logic was firstly formulated in a clear form by Leibniz. The initial results of mathematical logic originated from A. de Morgan (1806-1876) and G. Boole (1815-1864), whereas the entire later development goes back to Boole. His successors, W. S. Jevons (1835-1882) and especially C. S. Peirce (1839-1914) enriched the young science. The various developments of his predecessors were systematically expanded and completed by E. Schröder in his *Lectures on the Algebra of Logic* (1890-1895), which represent a certain conclusion to the series of developments emanating from Boole. The next stage was reached when G. Frege published his *Begriffsschrift* (1879) and his *Grundgesetze der Arithmetik* (1893-1903) and G. Peano and his colleagues began with the publication of the *Formulaire de Mathematiques* (1894), in which all mathematical disciplines were to be presented in a logical calculus. The publication of *Principia Mathematica* (1910-1913) by A. N. Whitehead and B. Russell marks a high point of this development.

For Hilbert and Ackermann, the requirement for formality is a priority, which must be secured by the use of a transparent and exact logical notation. For instance, in more complicated statements with an expression "Φ", it is advisable to write "~(Φ)" for "~Φ", so that it is clear which part of the sentence is negated. The requirement for clarity is especially important in the use of logical constants. When we say:

> "A candidate in mathematics and physics must be particularly thorough in mathematics, or he must be particularly thorough in physics", we do not mean that we want to exclude particularly thorough knowledge in both subjects at the same time. The "or" is used here in the sense of the Latin "vel" ("or also"). But when we say, "You have to work for the exam or you won't pass," we mean that the two cases are mutually exclusive. The "or" is used here in the sense of the Latin "aut-aut" ("either-or") (Hilbert and Ackermann 1928, 4).

Furthermore, with the logical notation "Φ ↔ Ψ" (read "Φ equivalent to Ψ") we denote a statement that is arguably called the co-implication of "Φ" and "Ψ". The expression "Φ ↔ Ψ" is correct if and only if "Φ" and "Ψ" have the same truth value, i.e. if "Φ" and "Ψ" are both right or both wrong. By combining basic connectives, we can also express the exclusive "either-or". "Either Φ or Ψ" can be represented by "~(Φ ↔ Ψ)". "~(Φ ↔ Ψ)" is correct if and only if "(Φ ↔ Ψ)" is wrong. This is the case if and only if of the two statements "Φ" and "Ψ" one is correct and the other is incorrect.

The definition of the logical constants is the fundamental step to formality, as grounding of logical validity. From the definition of our basic connectives (logical constants), the correctness or falsehood of a linked statement is only dependent on the correctness or falsehood of the basic statements, but not more from its content.

The value of the formalist ideal becomes equally important for the analysis of advanced logical ideas, such as those of the predicate calculus. In the third chapter of their *Grundzüge,* Hilbert and Ackermann declared the inadequacy of the previous calculus for the formal expression of the restricted predicate calculus. They stated some examples of this inadequacy of the traditional logical calculus in cases of logical relations:

> "If there is a president of all clubs in a city, every club in the city has a president." Here, too, one recognises the purely logical character of the sentence to the fact that the content of the terms "President" and "Club of the City" is irrelevant for the correctness of the sentence. But we are unable to explain the characteristic of this sentence with the help of the traditional logic or with the help of class calculus. The reason for this is that this is not just about certain properties, but rather a relationship between several objects. The term "President" implies such a relationship, namely a relationship between a person and a club. However, we have so far no means to represent such relationships with symbolic expressions (Hilbert and Ackermann 1928, 65).

It is precisely the relationships that play the essential role in the logical structure of mathematics. But it is not just taking mathematics into account that makes for us necessary to introduce a logic of relationships, like that first example shows. A second example:

> "x as a man has y as a child" is just another expression for "x is the father of y". On the other hand, the same sentence can also be pronounced like this: "y is the child of (man) x". The above sentence, hence, can only be put into evidence if "being father" and "being a child" is not primarily understood as a trait but as a relationship between two people, so that "x is father" and "x is child" are not the elementary sentences, but "x is father of y" and "x is child of y" (Hilbert and Ackermann 1928, 67).

The sentences "x is father" and "x is child" are then expressed, with the help of these elementary sentences, in the following way: "There is a y such that x is the father of y" and "There is a (man) y, so that the x is the child of y" - both of which sentences certainly indicate properties of x, but they are defined by a relation between two people. Since our previous calculation has turned out to be inadequate, we are compelled to seek how to expand our logical symbolism. Let us return to the point of our considerations, where we went beyond the propositional calculus. We saw there that many statements consist of a property assigned to an object. We have not fully exploited this decomposition of the propositions statements, by introducing indeed the properties (in the form of the associated classes) into the calculus, but not the objects. In this regard, Hilbert and Ackermann try to complete the calculus by providing statements which separate the objects (individuals) from the properties predicated (predicates) to them and explicitly designate both.

6. From Hilbert to Gentzen

The basic work of Gerhard Gentzen, his dissertation *Untersuchungen über das logische Schließen,* related to the area of predicate logic, which was also called "restricted function calculus" by Hilbert and Ackermann. In addition to classical logic, Gentzen deals also with intuitionistic logic, as it had been formalised by Heyting.

The formalisation of logical reasoning, as it was developed in particular by Frege, Russell and Hilbert, is quite a long way distanced from the kind of reasoning that was actually practiced in mathematical proofs, as Gentzen stated. For this reason, the main principle (Hauptsatz) of Gentzen's investigation

> implies that every purely logical proof can be reduced to a certain, by the way by no means unambiguous, normal form. The most essential properties of such a normal proof can be expressed as follows: It does not make any detours. No terms are introduced into it, which are not contained in its end-result and therefore have to be used in order to obtain it (Gentzen 1935a, 177).

The essential external difference between natural-intuitionistic (NJ) derivations and derivations in the systems of Russell, Hilbert, Heyting is as follows: In the latter, the correct formulas are derived from a series of "logical basic formulas" by a few conclusions; the natural inference, however, does not generally start from logical principles, but from assumptions ... to which logical conclusions follow (Gentzen 1935a, 184).

A natural-intuitionistic derivation consists of formulas arranged in the form of a family tree. The initial formulas of the derivation are assumption-formulas, each of which is assigned to exactly one derivational deductive figure and it then stands "above" its sub-formula. All formulas under an assumption formula, but still above the sub-formula of the derivational deductive figure, to which the assumption formula belongs, including also the assumption formula itself, are called dependent on it. The conclusion yet makes the statements that follow after it independent of the assumption belonging to it.

There are two rules: introduction (E: Einführung in German) and elimination (B: Beseitigung in German). The designations of the individual deductive figure schemes: UE, UA etc. should mean: deductive figure formed according to the scheme is an And-(U), Or-(O), All-(A), There-is-(E), Follow-(F) or non-(N)-sign "introduction" (E) or "elimination" (B). The deductive figure schemes are the following (Gentzen 1935a, 186):

	"introduction" (E)	*"elimination" (B)*
And-(U)	UE $\dfrac{\mathfrak{A} \quad \mathfrak{B}}{\mathfrak{A} \& \mathfrak{B}}$	UB $\dfrac{\mathfrak{A} \& \mathfrak{B}}{\mathfrak{A}} \quad \dfrac{\mathfrak{A} \& \mathfrak{B}}{\mathfrak{B}}$
Or-(O)	OE $\dfrac{\mathfrak{A}}{\mathfrak{A} \vee \mathfrak{B}} \quad \dfrac{\mathfrak{B}}{\mathfrak{A} \vee \mathfrak{B}}$	OB $\dfrac{\mathfrak{A} \vee \mathfrak{B} \quad \overset{[\mathfrak{A}]}{\mathfrak{C}} \quad \overset{[\mathfrak{B}]}{\mathfrak{C}}}{\mathfrak{C}}$
All-(A)	AE $\dfrac{\mathfrak{F}\mathfrak{a}}{\forall \mathfrak{x} \mathfrak{F}\mathfrak{x}}$	AB $\dfrac{\forall \mathfrak{x} \mathfrak{F}\mathfrak{x}}{\mathfrak{F}\mathfrak{a}}$
There-is-(E)	EE $\dfrac{\mathfrak{F}\mathfrak{a}}{\exists \mathfrak{x} \mathfrak{F}\mathfrak{x}}$	EB $\dfrac{\exists \mathfrak{x} \mathfrak{F}\mathfrak{x} \quad \overset{[\mathfrak{F}\mathfrak{a}]}{\mathfrak{C}}}{\mathfrak{C}}$
Follow-(F)	FE $\dfrac{\overset{[\mathfrak{A}]}{\mathfrak{B}}}{\mathfrak{A} \supset \mathfrak{B}}$	FB $\dfrac{\mathfrak{A} \quad \mathfrak{A} \supset \mathfrak{B}}{\mathfrak{B}}$
non-(N)	NE $\dfrac{\overset{[\mathfrak{A}]}{\curlywedge}}{\neg \mathfrak{A}}$	NB $\dfrac{\mathfrak{A} \quad \neg \mathfrak{A}}{\curlywedge}$
	$\dfrac{\curlywedge}{\mathfrak{D}}$	

We denote the free object variable designated in the relevant scheme with $\mathfrak{a}$ as the eigenvariable of an AE or EB. (Provided that it exists, i.e. that it comes out from the bound object variable labelled with $\mathfrak{x}$ in the formula labelled with $\mathfrak{F}\mathfrak{x}$). The eigenvariable of an AE must not appear in the formula marked with $\forall \mathfrak{x} \mathfrak{F}\mathfrak{x}$ in the scheme and not in any assumption formula on which it depends. The eigenvariable of an EB must not appear in the formula marked with $\exists \mathfrak{x} \mathfrak{F}\mathfrak{x}$ in the scheme, neither in the upper formula marked with $\mathfrak{C}$, nor in any assumption formula on which it depends, except for the assumption formulas designated with $\mathfrak{F}\mathfrak{a}$ belonging to the EB.

The meaning of the content of the NJ deductive figures

At the next stage, Gentzen tried to explain the meaning of some of the deductive figures and thus to make it clear that the calculation does indeed conform to the "real reasoning".

FE: If $\mathfrak{B}$ is proven using the assumption $\mathfrak{A}$, then (now without this assumption): from $\mathfrak{A}$ follows $\mathfrak{B}$. Of course, further assumptions may have been made, of which this result is still pending.

OB: ("case distinction"). If one has proven $\mathfrak{A} \vee \mathfrak{B}$, one can make a case distinction: one first assumes that it holds $\mathfrak{A}$ and derives something like $\mathfrak{E}$ from it. If, furthermore, $\mathfrak{E}$ can also be derived from the assumption of the validity of $\mathfrak{B}$, then $\mathfrak{E}$ holds at all, i.e. now independent of both assumptions.

AE: If $\mathfrak{F}\mathfrak{a}$ is proven for "any $\mathfrak{a}$", then $\forall\mathfrak{x}\mathfrak{F}\mathfrak{x}$ holds. The requirement that $\mathfrak{a}$ is "completely arbitrary" can be expressed more precisely as follows: $\mathfrak{F}\mathfrak{a}$ must not depend on any assumption in which the object variable $\mathfrak{a}$ occurs. And this, together with the natural requirement that in $\mathfrak{F}\mathfrak{a}$ the $\mathfrak{a}$ of $\mathfrak{F}\mathfrak{a}$ must be replaced by $\mathfrak{x}$ wherever it occurred, is precisely the part of the above "variable condition" relating to the AE.

EB: You have $\exists\mathfrak{x}\mathfrak{F}\mathfrak{x}$. Then one says: Let $\mathfrak{a}$ be such an object for which $\mathfrak{F}$ holds. Hence, one assumes: It is valid that $\mathfrak{F}\mathfrak{a}$. (Of course one has to take an object variable for $\mathfrak{a}$ that did not appear in $\exists\mathfrak{x}\mathfrak{F}\mathfrak{x}$). If, on the basis of this assumption, one has proven a statement $\mathfrak{a}$ which no longer contains and does not depend on any other assumption containing $\mathfrak{a}$, then $\mathfrak{E}$ is proven independently of the assumption $\mathfrak{F}\mathfrak{a}$. With what was said, the part of the "variable condition" relating to the EB was expressed. There is a certain analogy between EB and OB, just as the there-is-sign is the generalisation of the $\vee$, and the universal sign that of the &.

NB: $\mathfrak{A}$ and $\neg\mathfrak{A}$ mean a contradiction, and such a contradiction cannot be valid (theorem of contradiction). This is formally represented by the final figure NB, where $\curlywedge$ denotes "the contradiction", the "wrong".

NE: (reductio ad absurdum). If something wrong ($\curlywedge$) follows from an assumption $\mathfrak{A}$, then $\mathfrak{A}$ is not correct, i.e. it is the case that $\neg\mathfrak{A}$.

The scheme $\curlywedge$ / D: If there is something wrong, any statement is valid.

The advantage of the natural deduction is the achieved formality of the structural reasoning rules, the avoidance of superfluous detours and the sufficient emphasis on grounding. Moreover, Gentzen (1935a, p. 190-210) provided an alternative method to natural deduction, which he called sequent calculus (L calculus). Every line of a proof in sequent calculus is a conditional tautology (or theorem) with zero or more conditions on the left. Sequent calculus does not start from assumptions as natural deduction, but from logical formulas, in the manner of Logicism. It maintains however the division of reasoning forms into introductions and eliminations of the individual logical signs. The idea of sequent calculi and of structural rules was firstly elaborated by Hertz, as it was openly acknowledged by Gentzen (1933).

The natural deduction trees deriving from Gentzen's N calculus and the sequent-based systems originating in Gentzen's L calculus belong to the main proof systems used for the instruction of Logic. However, the most popular proof system for the presentation of natural deduction was constructed by Fitch, based on Jaśkowski's notation.

Endnotes:

1. Albert of Saxony (1522, IV 1, 24rb) followed Buridan's definition.
2. The distinction between natural and accidental or non-natural consequence is also found in Boethius (1969, 835B), Duns Scotus (1966, I, d. 11, q. 2, 136-137) and William of Sherwood (1941, 80).
3. Ockham adds that the rule: "Whatever follows to the antecedent follows to the consequent" is false.

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